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Interacting Gravitational and Dirac Fields With Torsion

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A general interaction scheme is formulated in a general space–time with torsion from the action principle by considering the gravitational, the Dirac, and the torsion field as independent fields. Some components of the torsion field come out to be automatically zero. Both the resulting Einstein-like and the Dirac-like fields equations contain nonlinear terms given by a self-interaction of the Dirac spinor and originally produced by torsion. The theory is specialized to the Robertson–Walker space–time without torsion. To solve he corresponding equations, that still have a complex structure, the spin coefficients have to be calculated explicitly from the tetrad employed. A solution, even if simple and elementary, is then determined.

KEY WORDS: gravity; Dirac field; torsion; R-W space–time.

1. INTRODUCTION

The study of the interaction between gravitation and matter field with spin and torsion has been always an argument of interest (Hehl *et al.*, 1976). Torsion gives additional degrees of freedom that can be used to extend the theory. Accordingly, modifications of both the Standard Model (Dobado and Maroto, 1996) as well as string theory (Hammond, 2000) have been done to include torsion. Recently many other aspects of torsion theory have been reviewed and the possibility of the true existence of the space–time torsion and of its detection has been considered (Shapiro, 2002). The effect of torsion on wave equations coupled to gravity is generally that of adding nonlinear terms to the conventional field equations (Hehl *et al.*, 1976). This is an aspect that has been recently taken into account in evaluating the neutrino oscillations (Alimohammadi and Shariati, 1999; Zhang, 2000). Also, the perturbation of the energy levels of the Hydrogen atom due to the presence of torsion has been evaluated (Zecca, 2002b). In these last mentioned papers the cosmological background is in some sense taken fixed because it is plausible to neglect the back reaction of a microscopic system on the universe.

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The consideration of torsion is relevant also in the formulation of cosmological models. In the well known Einstein–Cartan–Sciama–Kibble theory (e.g., Hehl *et al.*, 1976; Penrose and Rindler, 1984), torsion is directly connected to spin density of matter. However, the Einstein equations can be extended to describe cosmological models also by symmetry ansatz on the torsion tensor (Minkowski, 1986; Tsamparlis, 1979).

In considering cosmological models, it seems of interest however a theory where also the role of matter field appears explicitly (Hehl *et al.*, 1976). It is the object of this paper to reconsider the interaction of the gravitational and Dirac fields in a general space–time with torsion in the line of a previous paper (Zecca, 2002a). The Einstein and Dirac equations are deduced from a general action principle. The calculations are developed by preliminary separating the contorsion tensor in a standard way and considering the gravitational, the Dirac, and the torsion fields as independent fields. There follows on one hand the vanishing of some component of the torsion tensor. On the other hand both the coupled resulting Einstein and Dirac equations contain nonlinear self-interactions of the Dirac spinor produced by torsion. The solution of these equations seems difficult to be obtained, even in simplified example of space–time and only some general comments are possible.

In the last section we try to solve the equations in the torsion-free case in the context of the Robertson–Walker space–time. Also, in this scheme a solution of the equations is not easy. To obtain results, one is forced to go back to explicitly calculate the spin coefficients associated to the tetrad frame there employed. With the obtained values of the spin coefficients it is possible to calculate the spin connection components whose structure finally suggests a solution of the equations. The solution determined is elementary and holds for the open case of the Robertson–Walker space–time and for massless particles. A complete solution of the equations seems to require a more complex study.

2. PRELIMINARY ASSUMPTIONS

The framework of the following considerations is represented by a four-dimensional Lorentz manifold of metric $g_{\mu\nu}$ endowed with an affine connection ∇, whose coefficients $\Gamma^{\lambda}_{\mu\nu}$ fulfill the metric compatibility assumption: $\partial_\lambda g_{\mu\nu} - \Gamma^{\kappa}_{\lambda\mu} g_{\kappa\nu} - \Gamma^{\kappa}_{\lambda\nu} g_{\mu\kappa} = 0$. (For definitions and mathematical conventions we refer to Nakahara, 1990; see also Zecca, 2002a). The affine coefficients can be expressed in terms of the Christoffel $\{{}^{\kappa}_{\mu\nu}\}$ coefficient (whose associated Levi–Civita connection is denoted by $\tilde{\nabla}$) and of the contorsion tensor $K^{\kappa}_{\cdot\mu\nu}$ as: $\Gamma^{\kappa}_{\mu\nu} = \{{}^{\kappa}_{\mu\nu}\} + K^{\kappa}_{\cdot\mu\nu}(K_{\lambda\mu\nu} = -K_{\nu\mu\lambda})$. By setting $\tau_\mu = g^{\alpha\beta} K_{\alpha\beta\mu}$, $3\mathcal{A}^\sigma = \epsilon^{\sigma\alpha\beta\mu} K_{\alpha\beta\mu}$, the contorsion tensor can be decomposed into the form $6K_{\alpha\mu\nu} = 2(g_{\alpha\mu}\tau_\nu - g_{\nu\mu}\tau_\alpha) + 3\mathcal{A}^\sigma \epsilon_{\sigma\alpha\mu\nu} + 6U_{\alpha\mu\nu}$. This relation indeed defines $U_{\alpha\mu\nu}(U_{\alpha\mu\nu} = -U_{\nu\mu\alpha}, g^{\alpha\mu} U_{\alpha\mu\nu} = 0, \epsilon^{\sigma\alpha\mu\nu} U_{\alpha\mu\nu} = 0)$ (Buchbinder *et al.*, 1992; Shapiro, 2002; Zecca, 2002a).

By the above decompositions, the Riemann tensor $R^{k}_{\cdot\lambda\mu\nu}$, the Ricci tensor $R_{\mu\nu} = R^{k}_{\cdot\lambda k\nu}$ and the scalar curvature $R = g^{\mu\nu}R_{\mu\nu}$ can in turn be decomposed into a part containing and in a part not containing the contorsion tensor. It is then possible to give an explicit expression of the gravitational (Einstein–Cartan) Lagrangian $L_g = -\sqrt{g}R$, $(g = |\det g_{\mu\nu}|)$. Apart from a divergence term that will be neglected because no variations of the boundary will be considered, one gets (e.g., Alimohammadi and Shariati, 1999; Zecca, 2002a)

$$L_g = -\sqrt{g}\left(\tilde{R} - \frac{1}{3}\tau^2 + \frac{3}{2}\mathcal{A}^2 + U_{\alpha\mu\nu}U^{\mu\alpha\nu}\right) \tag{1}$$

where $\tilde{R}$ is the usual scalar curvature expressed in terms of the Christoffel symbols.

The object of the following sections is to describe the interaction of the gravitational and Dirac fields in the general scheme with an a priori nontrivial contorsion. The description of the Dirac spinor ψ in the context of the four-dimensional Lorentz manifold is done by considering a Lagrangian both invariant under Lorentz rotations and scalar coordinate change. (Nakahara, 1990; Weinberg, 1972). This involves the consideration of a local reference frame e_a^μ (tetrad) defined by $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$, $(\eta_{ab} = e_a^\mu\, e_b^\nu g_{\mu\nu})$, $\eta_{ab} = \text{diag}\,(1, -1, -1, -1)$ being the Minkowski metric (Latin letters label tetrad vectors, greek latters label coordinate indices). From the Dirac matrices γ^a, satisfying $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, one defines the matrices $\gamma^\mu = e_a^\mu \gamma^a$. They satisfy the relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. The mentioned Lagrangian is then expressed by (Nakahara, 1990)

$$L_d = \sqrt{g}\bar{\psi}[i\gamma^\mu(\partial_\mu + \Omega_\mu) + m]\psi \tag{2}$$

where $\bar{\psi} = \psi^+\gamma^0$ and the spin connection Ω_μ is defined through

$$\gamma^\mu\Omega_\mu = -\frac{1}{8}e_a^\mu\, e_b^\nu(\nabla_\mu e_{c\nu})\gamma^a[\gamma^b, \gamma^c]. \tag{3}$$

Also, the spin connection can be separated in a part containing and in a part not containing the contorsion tensor. By proceeding as in Zecca (2002a) one obtains

$$L_d = \sqrt{g}\bar{\psi}\left\{i\gamma^\mu\left[\partial_\mu + i\gamma^5\left(A^g_{1\mu} + A^t_{1\mu}\right) + A^g_{2\mu} + A^t_{2\mu}\right] + m\right\}\psi \tag{4}$$

where

$$\begin{aligned}
A^g_{1\mu} &= \frac{1}{4}e_{b\mu}(\partial_\sigma e_{c\nu} - \partial_\nu e_{c\sigma})\epsilon^{bcad}e_a^\nu\, e_d^\sigma \\
A^g_{2\mu} &= -\frac{1}{2}e_\mu^c\tilde{\nabla}_\alpha e_c^\alpha \\
A^t_{1\mu} &= \frac{3}{4}\mathcal{A}_\mu, \qquad A^t_{2\mu} = \frac{1}{4}\tau_\mu
\end{aligned} \tag{5}$$

By considering the spin coefficients (Ricci rotation coefficients: e.g., Chandrasekhar, 1983; Penrose and Rindler, 1984) defined by $\gamma_{abc} = e_a^\nu(\tilde{\nabla}_\mu e_{b\nu})e_c^\mu$, the contorsion independent part of the spin connection can also be expressed as (Zecca, 2002a)

$$A^g_{1\mu} = -\frac{1}{4}\epsilon^{dabc}\gamma_{bca}\, e_{d\mu}, \qquad A^g_{2\mu} = -\frac{1}{2}\gamma^{ad}_{\cdot\cdot a}\, e_{d\mu}. \tag{6}$$

By using the definition of the gamma matrices, the expression (3) can be worked out in a different way to obtain $\Omega_\mu = -\frac{1}{4}\gamma^\nu\nabla_\mu\gamma_\nu$. Therefore the Dirac Lagrangian can alternatively be written as

$$L_d = \sqrt{g}\bar{\psi}\left[i\gamma^\mu\left(\partial_\mu - \frac{1}{4}\gamma^\nu\nabla_\mu\gamma_\nu\right) + m\right]\psi \tag{7}$$

where, however, the contorsion-dependent term is not immediatly separated.

3. INTERACTING GRAVITATIONAL AND DIRAC FIELDS WITH TORSION

The object is now to describe the interacting gravitational and Dirac fields from the general Lagrangian $L = L_d + L_g$, with L_g and L_d given in (1) and (2). The field equations are obtained by applying the general Euler–Lagrange equation $\partial_L/\partial_\eta - \tilde{\nabla}_\alpha(\partial L/\partial\tilde{\nabla}_{\alpha\eta}) = 0$. By choosing η to be anyone of the fields $U^{\alpha\beta\gamma}$, τ^μ, $\mathcal{A}^\mu$, $\bar{\psi}$ considered as independent fields and by using the expression (4) of L_d, one obtains respectively

$$U_{\alpha\beta\gamma} = 0 \tag{8}$$

$$\tau_\mu = -\frac{3}{8}i\bar{\psi}\gamma_\mu\psi \tag{9}$$

$$\mathcal{A}_\nu = -\frac{1}{4}\bar{\psi}\gamma_\nu\gamma^5\psi \tag{10}$$

$$\gamma^\mu\big[\partial + i\gamma^5\big(A^g_{1\mu} + A^t_{1\mu}\big) + A^g_{2\mu} + A^t_{2\mu}\big]\psi = im\psi. \tag{11}$$

When the expressions (5) are inserted into Eq. (11) by the use of the results (9) and (10), one gets a nonlinear equation that can as well be called Dirac equation. Nonlinearity is given by self-interaction of Dirac spinor produced by torsion. To obtain the analog of the Einstein field equations it is useful to proceed by steps.

By first varying L_g with respect to $g_{\mu\nu}$ one obtains.

$$\begin{aligned}\frac{\partial L_g}{g_{\mu\nu}} = \sqrt{g}\bigg[&\tilde{R}^{\mu\nu} - \frac{1}{2}\tilde{R}g^{\mu\nu}\\ &+ \frac{1}{3}\left(\frac{1}{2}g^{\mu\nu}\tau^2 - \tau^\mu\tau^\nu\right) - \frac{3}{2}\left(\frac{1}{2}g^{\mu\nu}\mathcal{A}^2 - \mathcal{A}^\mu\mathcal{A}^\nu\right)\\ &- \frac{1}{2}g^{\mu\nu}U_{\alpha\beta\gamma}U^{\beta\gamma\alpha} - U^\nu_{\cdot\alpha\beta}U^{\alpha\mu\beta} - U^{\nu}_{\alpha\beta\cdot}U^{\alpha\beta\mu} - U^{\mu\alpha\beta}U^\nu_{\alpha\cdot\beta}\bigg],\end{aligned} \tag{12}$$

where standard results on the variations of $g^{\mu\nu}$, g, $\tilde{R}_{\mu\nu}$ with respect to $g_{\mu\nu}$ have been employed (e.g., Nakahara, 1990). Secondly, to calculate the variation of L_d with respect to $g_{\mu\nu}$, it is useful to take into account the following results (Round brackets mean symmetrization; see also Shapiro, 2002):

$$\frac{\partial e^b_\beta}{\partial g_{\mu\nu}} = \frac{1}{2} e^{b(\mu} \delta^{\nu)}_\beta, \qquad \frac{\partial e^\lambda_b}{g_{\mu\nu}} = -\frac{1}{2} g^{\lambda(\mu} e^{\nu)}_b \tag{13}$$

$$\frac{\partial \nabla_\alpha \gamma_\beta}{\partial g_{\mu\nu}} = \nabla_\alpha \frac{\partial \gamma_\beta}{\partial g_{\mu\nu}} = \frac{1}{2} \delta^{(\nu}_\beta \nabla_\alpha \gamma^{\mu)}. \tag{14}$$

By using the Dirac Lagrangian in the form (7) one then obtains

$$\begin{aligned} \frac{\partial L_d}{\partial g_{\mu\nu}} &= \frac{1}{2}\sqrt{g} g^{\mu\nu} \bar{\psi}[i\gamma^\alpha(\partial_\alpha + \Omega_\alpha) + m]\psi + \frac{i}{2}\sqrt{g}\bar{\psi}\left[\gamma^{(\mu}\partial^{\nu)} + \gamma^{(\mu}\Omega^{\nu)}\right]\psi \\ &= +\frac{i}{2}\sqrt{g}\bar{\psi}\left[\gamma^{(\mu}\partial^{\nu)} + \gamma^{(\mu}\Omega^{\nu)}\right]\psi. \end{aligned} \tag{15}$$

The last equality follows because the first-term in square brackets in (5) vanishes as it can be checked by deriving the Dirac equation from the Lagrangian (2). (See the following Eq. (17)). [One can check that the result after the first equality in (15) is similar to the one that would be obtained by varying only g and $g^{\mu\nu}$ (but not the other terms), in the Dirac Lagragngian written as $L_d = \sqrt{g}\bar{\psi}[ig^{\mu\nu}\gamma_\nu(\partial_\mu + \Omega_\mu) + m]\psi$.]

By combining the previous results, one is left with the coupled Einstein and Dirac equations:

$$\begin{aligned} \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}g_{\mu\nu} + \frac{1}{3}\left(\frac{1}{2}g^{\mu\nu}\tau^2 - \tau^\mu\tau^\nu\right) - \frac{3}{2}\left(\frac{1}{2}g^{\mu\nu}\mathcal{A}^2 - \mathcal{A}^\mu\mathcal{A}^\nu\right) \\ = -\frac{i}{2}\bar{\psi}\left[\gamma_{(\mu}\partial_{\nu)} + \gamma_{(\mu}\Omega_{\nu)}\right]\psi \end{aligned} \tag{16}$$

$$\gamma^\mu(\partial_\mu + \Omega_\mu)\psi = im\psi, \quad \left(\Omega_\mu = -\frac{1}{4}\gamma^\nu\nabla_\mu\gamma_\nu\right). \tag{17}$$

The Dirac equation (17) is nonlinear since Ω_μ depends on ψ in the way pointed out in correspondence to Eq. (11). By contracting the indices in Eq. (16) and using Eq. (17) one obtains

$$\tilde{R} - \frac{1}{3}\tau^2 + \frac{3}{2}\mathcal{A}^2 = \frac{m}{2}\bar{\psi}\psi, \tag{18}$$

a relation that holds in general and that in absence of torsion gives $\tilde{R} = \frac{m}{2}\bar{\psi}\psi$. Therefore in absence of torsion and for massless particles the Ricci scalar vanishes. It seems difficult to give other properties or solutions of the coupled equations (16) and (17) even in simplified examples of space–time. It is possible to make some considerations on the equations in case of the Robertson–Walker space–time and in absence of torsion.

4. TORSION-FREE INTERACTING GRAVITATIONAL AND DIRAC FIELDS IN ROBERTSON–WALKER SPACE–TIME

The previous scheme is now specialized to the Robertson–Walker space–time of metric

$$ds^2 = dt^2 - R(t)^2\left[\frac{dr^2}{1-ar^2} + r^2(d\theta^2 + \sin\theta^2 d\varphi^2)\right], \quad a = 0, \quad \pm 1. \tag{19}$$

It is possile to choose for e_a^μ the (null) tetrad frame

$$\begin{aligned} e_{1\mu} &= \frac{1}{\sqrt{2}}\left(1, -\frac{R}{\sqrt{1-ar^2}}, 0, 0\right) \quad & e_{2\mu} &= \frac{1}{\sqrt{2}}\left(1, -\frac{R}{\sqrt{1-ar^2}}, 0, 0\right) \\ e_{3\mu} &= \frac{rR}{\sqrt{2}}(0, 0, -1 - i\ \sin\theta) & e_{4\mu} &= \frac{rR}{\sqrt{2}}(0, 0, -1, i\ \sin\theta) \end{aligned} \tag{20}$$

from which the gamma matrices γ^α's can be constructed in a standard way (Penrose and Rindler, 1984). Accordingly, by recalling the explicit expressions of the Ricci tensor and the Ricci scalar (e.g., Kolb and Turner, 1990), by taking into account the results (4), (5), and (6) and by neglecting contorsion, the Eqs. (16) and (17) read now

$$\bar{\psi}[\gamma_{(\mu}\partial_{\nu)} + \gamma_{(\mu}\Omega_{\nu)}]\psi = 0, \quad (\mu \neq \nu) \tag{21}$$

$$\frac{\dot{R}^2}{R^2} + \frac{a}{R^2} = \frac{i}{6}\bar{\psi}[\gamma_t(\partial_t + \Omega_t)]\psi, \quad (\mu = \nu = t) \tag{22}$$

$$\left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{a}{R^2}\right)g_{kk} = \frac{i}{2}\bar{\psi}[\gamma_k(\partial_k + \Omega_k)]\psi, \quad (\mu = \nu = k = r, \theta, \varphi) \tag{23}$$

$$\gamma^\mu(\partial_\mu + \Omega_\mu)\psi = im\psi, \quad \Omega_\mu = -\left(\frac{1}{4}\epsilon^{dabc}\gamma_{bca} + \frac{1}{2}\gamma^{ad}_{\ \ ..a}\right)e_{d\mu}. \tag{24}$$

The solution of the system (21)–(24) is still difficult and one can only conclude that for massive particles

$$\partial_k(\bar{\psi}\psi) = 0 \quad k = r, \theta, \varphi \tag{25}$$

that is $\bar{\psi}\psi$ is allowed to depend only on t in accordance with Eq. (18) because $\tilde{R}$ depends only on t in the present case. Therefore an explicit calculation of Ω_μ seems to be unavoidable. To that hand, we recall that the spin coefficients γ_{abc} can be calculated from the tetrad by means of the formulas (e.g., Chandrasekhar, 1983):

$$\gamma_{abc} = \frac{1}{2}[\lambda_{abc} + \lambda_{cab} + \lambda_{bca}],$$
$$\lambda_{abc} = e_{b\mu,\nu}\left[e_a^\mu\, e_c^\nu - e_a^\nu\, e_c^\mu\right]. \tag{26}$$

Many spin coefficients were obtained in Montaldi and Zecca (1994). From those coefficients one can construct Ω_μ as in (24). One finally arrives at the expressions:

$$\Omega_0 = -\frac{3}{2}\frac{\dot{R}}{R}; \qquad \Omega_r = -\frac{1}{r}; \qquad \Omega_\theta = -\frac{1}{2}\cot\theta; \quad \Omega_\varphi = 0. \tag{27}$$

(The contribution to Ω_μ of the first term in Eq. (6) comes out to be zero.) The previous considerations suggest, in case of massless particles, to look for solutions of the form $\psi = r(\sin\theta)^{1/2} f(t)\psi_0$ where ψ_0 is a constant four-dimensional spinor and $f(t)$ is a to be determined function of t. The mentioned spinor satisfies $(\partial_k + \Omega_k)\psi = 0$, $k = r, \theta, \varphi$. In order $(\partial_t + \Omega_t)\psi = 0$ it must be $f \sim R^{3/2}$ and it is then possible to satisfy also Eqs. (22) and (23) with $R =$ const if $a = 0$ and with $R = t + c_0$ if $a = 1$ (c_0 is an integration constant). Therefore

$$\psi = r(\sin\theta)^{1/2}(t + c_0)^{3/2}\psi_0 \tag{28}$$

is a solution of the Einstein Dirac equations for massless particles in the open universe case of the Robertson–Walker space–time. It remains open the problem of finding other solutions to the problem. This would be both of mathematical interest and of cosmological relevance.

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